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PREFERENCE ANALYSIS:

A General Method with Application to

The Cost of Living Index

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A deck of program cards, for the computer program set out in Section 9, for the method described in Section 8 of ranging a cost-of-living index relative to the quadratic preference hypotheses admissible on expenditure data for four occasions, punched according to the Bell II system for the IBM 650 Computer, may be obtained on request to the Econometric Research Program, Princeton University.

Introduction.

The value of money is equated with what can be got with it, and this depends on the prices. The consequent dependence of the value of money on prices, and how this can be practically investigated, is what is here for discussion. Genesis is in the form of question: Which is better - to spend a certain amount of money at certain prices, or another amount, at some other prices? The question is vague, though indispensible. Everything depends on the manner in which it is to be made intelligible. But when this manner is decided, in certain conspicuous possible ways, it will appear as a perfectly definite and answerable question.

What can be got by expenditures with a certain amount of money at certain prices is any combination of goods, to be called a composition, whose cost at those prices does not exceed that amount of money. This restriction on composition, defining a balance, depends jointly on the prices and the available money expenditure, but no more than on the ratio of prices to that expenditure.

A composition is said to be within, on or over a balance, determined by prices and an expenditure, according as its cost at those prices is at most, equal to, or greater than that expenditure. The conjunctions of balance and composition found by observation, each defining an expenditure figure, and together defining an expenditure configuration, are to be the data for the question.

Moreover, the question is to be made intelligible in terms of an hypothetical preference system which determines a best composition on any balance, which will also be the best composition within that balance. Such a preference system will be called a proper preference system. It is admissible in respect to any observed balance with a composition on it if that composition is, according to the system, the best one in the balance. When preference is measured by a differentiable preference function, with direction defined by the direction of the gradient, this criterion is Gossen's law, that price and preference directions coincide in equilibrium.

However, if any data thus admits a proper preference system as an hypothesis, by which it will be said to be consistent, it must admit an infinity of such hypotheses. Preference analysis of the data is to be an analysis of such a class of hypotheses, which is either empty or infinite.

The fundamental question can now be put more explicitly: Which is better, the best composition of goods that can be got with one expenditure at one set of prices, or with another expenditure at another set of prices? The decision of the better and the best here is to be according to some preference hypothesis admissible on given data.

This question underlies the measurement problem of the cost of living. On any occasion expenditure is made at certain prices to obtain goods desired in the process of living, supposedly in such a manner as to achieve the highest standard attainable with the expenditure at the prices. If prices change, then expenditure has to change compensatingly, if the same standard is to be maintained. To specify this compensating change, the cost of living index, with one occasion as base and another as current occasion, is defined as the multiplier of expenditures which will compensate it for the price-change from the base to the current occasion, according to some hypothetical preference system. But if this preference system is free to range in the admissible class for some data, this index will be free to describe a certain range. A problem, now presented, and to which a solution will be given, is to determine the range of a cost of living measurement thus prescribed by given data.

Samuelson[1] has pointed out the indeterminacy of the cost of living measurement; but that leaves the question of determining the extent of the indeterminacy. Some formulae which have been obtained achieve this determination; but they are difficult to handle, and no numerical work has yet been done with them. However, in the familiar case of making the cost of living measurement between two periods, on the basis of expenditure data just for those periods, the answer is plain: The conventional index gives an upper limit, which can be approached arbitrarily closely. That is, no better limit can be obtained without more data, or without more restrictive assumptions about the underlying preference structure beyond that it is of the normal type. This is the most important fact about index-number theory with two-period data; that is, the possibility of having developed a theory, thus confined, is almost non-existent, and this is confirmed historically. But by allowing more data, development, as has been exemplified by the method of Wald,[2] is possible.

Wald has considered preference systems on a quadratic model, represented by a quadratic function of composition whose magnitudes decide the preference relation between compositions. Now a quadratic function can represent a proper preference system in a region only if it is strictly convex and increasing in that region. This is because only a strictly convex quadratic can have strictly convex levels. Further, the Engel curves, which are the loci of composition as expenditure varies while prices are fixed, are lines concurrent in a point which defines the centre of such a system.

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[1]Paul A. Samuelson, Foundations of Economic Analysis (Harvard, 1947).

[2]A. Wald, "A New Formula for the Index of Cost of Living," Econometrica, Vol. 7, No. 4 (October, 1939), pp. 319-335.

Thus, not any pair of linear Engel curves can belong to a quadratic system: firstly, they have to be concurrent; and secondly, their intersection has to be in a point over the balances associated with the region in which the system is valid. However, these matters left aside, Wald has shown that if linear Engel curves corresponding to two sets of prices are known, then the cost of living measurement is determinate between occasions which have those prices, on the hypothesis of a quadratic model. He then shows that if these Engel curves pass through the origin, the determination reduces to that provided by Fisher's "ideal index."[3] But it should be noted that an admissible quadratic preference hypothesis cannot have its centre at the origin. Samuelson's revealed preference axiom[4] can be used to decide this. The justification of the Fisher index in the quadratic model which could arise here and which was proposed by Buscheguennce,[5] and Konüs[6] is altogether illegitimate. Far from there being formal arguments going in favour of the Fisher index, there are precise formal arguments refuting its validity.

Now consider an expenditure configuration of four figures, corresponding to four sets of prices. If the sets of prices are parallel in pairs, and a quadratic model is assumed, then the two lines joining corresponding compositions must be Engel curves, and we have the situation

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[3]Irving Fisher, The Making of Index Numbers. (New York: Houghton Mifflin Company, 1927).

[4]P. A. Samuelson, "Consumption theory in terms of revealed preference," Economica 28 (1948), pp. 243-53.

[5]S. S. Buscheguennce, "Sur une classe des hypersurfaces. A propos de 'l' index idéal' de M. Irv. Fisher," Recueil Mathématique, XXXII, 4 (1925), Moscow.

[6]A. A. Konüs, "The Problem of the True Index of the Cost of Living," The Economic Bulletin of the Institute of Economic Conjuncture, Moscow, No. 9-10 (36-37) (September-October, 1924), pp. 64-71.

with a pair of Engel curves hypothetically known of Wald. But again, these lines have to intersect in a point with a proper location for the quadratic hypothesis to be legitimate.

Wald's case can be considered a special case of a general configuration of four figures. This general case has been investigated and it is found that though the configuration may be absolutely consistent, in the sense of consistency already described, it need not be quadratically consistent, in the sense of including in the class of admissible hypotheses a subclass on the quadratic model. However, if the quadratic class exists, it will be infinite; and correspondingly, a cost of living measurement is ranged in an interval, rather than having the point determination usually sought, and which is obtained in Wald's special case. Point determination is also obtained in the case, intermediate between the general case and Wald's case, in which just one Engel curve is known, that being for the object occasion, and there is data for two other occasions, in which the prices need not be parallel. When these prices are parallel, there is return again to Wald's case.

The formulae obtained for ranging the cost of living on the quadratic preference model are bound up with the formulae for ranging the cost of living obtained without any further assumption on preference structure beyond that it is of the normal type: they are bound up in such a way as to exhibit them as providing a smoothing principle, selecting a subclass of admissible hypotheses which are distinguished as, in a certain sense, fitting the data most smoothly. The cost of living interval determined on the quadratic normal model thus has a statistical meaning, in relation to the larger interval determined just on the normal model, giving it a special importance.

A computer program has been prepared by Mr. Harold Samuels for the criteria for the validity of the quadratic model, and the cost of living calculations based on that model when it is admissible. It has been applied to a number of numerical examples, including one in two dimensions which admits a graphical representation.

1. Market choice

In the picture of the market, there is some finite number n of simple goods, which can be considered as the components $\mathcal{b}_1, \ldots, \mathcal{b}_n$ of a composite good

$$\mathcal{b} = \{\mathcal{b}_1, \ldots, \mathcal{b}_n\} .$$

These goods are available for purchase in any amounts at prices $\pi_1, \ldots, \pi_n$ given on any occasion, and forming a vector

$$p = \{\pi_1, \ldots, \pi_n\} .$$

A collection of amounts $\xi_1, \ldots, \xi_n$ of the different goods defines a composition in these goods, and is represented by a vector

$$x = \{\xi_1, \ldots, \xi_n\} .$$

The purchase (p,x), of a composition x at prices p, requires an expenditure

$$e = p'x ,$$

where

$$p'x = \pi_1\xi_1 + \ldots + \pi_n\xi_n$$

is the scalar product of the vector p with the vector x.

When the expenditure e is taken as the unit of money, prices p transform to relative prices

$$u = \frac{p}{e} ,$$

where $\frac{p}{e} = \{\frac{\pi_1}{e}, \ldots, \frac{\pi_n}{e}\}$. The vector u defines the <u>balance</u> determined by an expenditure e with prices p. A composition x is said to be <u>within</u>, <u>on</u>, or <u>over</u> a balance u according as

$$u'x \leq 1 , \; u'x = 1 \text{ or } u'x > 1 .$$

A purchase (p,x) has associated with it the balance $u = \frac{p}{e}$, where $e = p'x$; so that

$$u'x = 1 .$$

Thus the composition in a purchase is on the associated balance. The compositions y attainable at the same prices for no greater expenditure are those such that

$$u'y \leq 1 .$$

These compositions can be had as well as x without loss by further expenditure. But, in the purchase, x is singled out from among them.

The market thus appears as a field of choice to which money gives access. An amount e of money, when the prices are p, gives choice of goods in every composition x such that

$$u'x \leq 1 ,$$

where $u = \frac{p}{e}$.

The market of common experience is also a field of choice arising from opportunities of exchange of money for goods. But as such it can hardly be grasped in all its complexities within the picture that has just been described. However, this picture gives a basic form for approaching the market. But how difficult it is to apply the picture to reality is seen in the multiplicity and complexity of commodities, in their uncertain nature, and uncertain availability, and the difficulties in the very notion of price. And there are all kinds of other imponderabilities with local and temporal association to vitiate the picture besides.

2. Preference hypothesis.

Any act has the form of a choice. On any occasion, a single action is impelled, and what is done appears as a selection from the plural possibility of what might have been done, in other words a choice. Though there is no escape from this form, and an exploitation of it gives the general method now to be followed, the application of this form is usually no simple matter, as has been indicated in the case of the market.

When one possibility is selected by an agent out of an otherwise undifferentiated variety of possibilities, this is taken as a manifestation of preference for that selected possibility on the part of the agent. Thus, when a purchase (p,x) is made, this is taken to show the preference, on the part of the purchaser, for x over every other composition y within the balance associated with the purchase. Any other such composition is equally a candidate for choice, at no greater expenditure, but x is selected from among them; and the method of explanation is to suppose that this is because x is preferred to all these other compositions, in some hypothetical preference system, represented by an order relation S on the set of all compositions. Thus x is supposed elected, which is to say selected by preference.

Thus, from the purchase (p,x) is made the inference xSy for all $y \neq x$ such that $u'y \leq 1$, where $u = \frac{p}{e}$ and $e = p'x$. In this inference, prices p enter only to the extent of determining the balance u.

Such an order S of compositions, which thus regulates market choice, defines a preference hypothesis. It elects a composition x on a balance u as preferred to all other compositions within u.

In potentiality, u can be any vector with positive elements. A proper preference system is defined to be a minimal preference system which is sufficiently complete to elect a composition on <u>any</u> balance from all those compositions within that balance.

Thus, if P is a proper preference system, then it is an order with the property that to every balance u there corresponds a composition x, which may be denoted by $x = f_P(u)$, such that $u'x = 1$, and xPy for every $y \neq x$ such that $u'y \leq 1$.

3. Expenditure systems.

A functional dependence $x = f(u)$ of composition x on balance u such that $u'x = 1$, that is, which determines a composition on every balance, is to define an <u>expenditure system</u>. Generally, u and x can be considered confined to certain <u>balance</u> and <u>composition domains</u> B and C, each being a region in the positive orthant of a Cartesian space of dimension n. An expenditure system of the form $f = f_P$, that is, which is derived from some proper preference system P, is called a <u>proper expenditure system</u>.

Now if $f = f_P$ where P is a proper preference system, then it appears that $P = P_f$, where

$$P_f = \vec{Q}_f$$

is the chain extension (transitive closure) of the relation Q_f defined by

$$xQ_f y \equiv \bigvee_u x = f(u) \wedge y \neq x \wedge u'y \leq 1 .$$

It follows that f is proper if and only if P_f is an order; and since, in its construction, P_f is in any case transitive, all that need be asked is that it be irreflexive.

The proper preference systems are thus of the form $P = P_f$, where f is an expenditure system for which P_f is irreflexive.

Enquiry has to be made into the structure of proper preference systems. Generally, not much can be said very concisely. But if some further, possibly quite simple conditions are imposed, then the whole matter becomes quite concise, and familiar.

For instance, let it be asked, of an expenditure system f, that

$$\left|f\left(\frac{u}{\rho}\right) - f\left(\frac{u}{\rho}\right)\right| < M|\rho - \sigma|$$

by which condition (introduced by Houthakker[1]) f might be called <u>regular</u> (following Uzawa[2]). Now let a <u>regular preference system</u> be one of the form $P = P_f$ where f is a regular expenditure system. Then, for such a preference system P, the following can be said: it is a proper preference system for which there exists a numerical function $\varphi = \varphi(x)$, which is strictly increasing, whose levels are strictly convex, and has the property

$$\varphi(x) > \varphi(y) \iff xPy ,$$

by which it represents P. By this property, φ might be said to <u>measure</u> P, or to be a <u>gauge</u> for P. If φ is a gauge for P, then so is $\omega(\varphi)$, where $\omega(t)$ is any strictly increasing function. So the gauges of P form an infinite class of functions.

It is desirable to proceed only with the most essential suppositions. While a proper preference system appears as the essential concept, it will be seen that some further conditions to be imposed, like regularity, make for no new departure from the essential, when there is reference to given empirical, and therefore finite, data.

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[1] H. S. Houthakker, "Revealed preference and the utility function," <u>Economica</u> 17 (1950), pp. 159-174.

[2] H. Uzawa, "On the logical relation between preference and revealed preference," <u>Technical Report</u> No. 38, Department of Economics, Stanford University, 1956; or <u>Mathematical Methods in the Social Sciences</u>, Stanford Mathematical Studies in the Social Sciences V (Stanford, 1959).

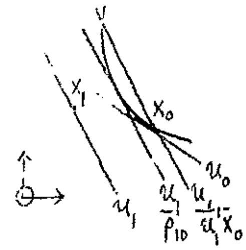
An expenditure system is the more basic concept, rather than a preference system; since it deals directly with acts of behaviour, rather than a controlling preference system. A regular preference system has the advantage that it is associated with a regular expenditure system, and also that it is representable by a numerical function which is strictly increasing and whose levels are strictly convex. Any such function will be called a <u>normal preference function</u>. While a regular preference system always has a normal preference function for a gauge, and any normal preference function is a gauge for a proper preference system, it is not obvious, nor yet proved, nor perhaps important, that a normal preference function should always be a gauge for a regular preference system. Regularity is just an example of a simple condition on behaviour which gives an admitted preference system a peculiarly simple structure.

It appears that a normal, for example, a regular, preference system has <u>levels</u>, defined by the levels of any function it has for a gauge; and these levels are completely ordered. Preference order for compositions on different levels corresponds to the order of those levels; and there is no preference order determined for compositions on the same level. Such a peculiar kind of order, which is represented by a complete order of classes of elements between which there is no order, defines a <u>scale</u>. Thus a regular preference system is a scale, though a proper preference system is not necessarily a scale. The relation measured by a normal preference function may be called a <u>normal preference scale</u>. It is apparently somewhat more general than a regular preference scale; it is a proper preference system with the additional requirement that it be a scale, and is something between a proper and a regular preference system, having generality together with a particularity that makes it peculiarly workable. But these distinctions, which are conspicuous for the formal concepts, lose substance in empirical application, when they appear to have equal scope.

4. Cost of living.

According to the idea of the cost of living, it is the cost of maintaining at given prices a given standard of living. Goods are consumed in the process of living and the different goods are consumed in different amounts which together define a possible composition of consumption. Standard of living is defined by preference between such possible compositions.

Thus cost of living is made intelligible in terms of preference between possible compositions of consumption, say according to some normal preference scale P. Suppose prices are fixed within certain periods, say years, and to vary between years, and that consumption is measured as the aggregate throughout a year. Consider two years, to be indicated by 0 and 1, and which are to have the role of a base and a current year. Consumption in these years appears with composition x_o, x_1 and is obtained at prices p_o, p_1 and therefore for expenditures $e_o = p_o{}'x_o$, $e_1 = p_1{}'x_1$; so the balances are $u_o = \frac{p_o}{e_o}$, $u_1 = \frac{p_1}{e_1}$. It is asked: what is the minimum cost e_{1o} at the current prices p_1 of maintaining a standard not inferior to that found in the base year? This minimum cost e_{1o} defines the cost of living, for the current year relative to the base year. The multiplier ρ_{1o}, such that $e_{1o} = e_1\rho_{1o}$, of current expenditure which will thus compensate for the price change between the base and current year, defines the cost-of-living index, for the current year relative to the base year. It is defined by the condition that $f_P(u_o)$ and $f_P(\frac{u_1}{\rho_{1o}})$ be on the same preference level.



It is all very well to have a cost-of-living concept depending on preferences. But preferences have to be known, if they are to be of any use; and generally they are not altogether known. They are only known to the extent that they can be determined by given data, necessarily finite, and therefore fragmentary. Thus, empirically, preferences can be determined only incompletely. Correspondingly, there will be an incomplete determinacy in the cost-of-living measurement. The problem which presents itself is to discover the extent to which it is determined, on any given data.

Hypothetical preferences can only be inferred from their hypothetical effects on choice, as expressed in expenditures. Hence, expenditure data for a variety of different occasions, say years, are to be made the basis for the measurement of the cost of living, in respect to the prices found in one of these years and the standard of living obtained in another. This is how the cost-of-living problem is first to be conceived. Eventually, it is some kind of statistical concept which is not invalidated, however erratic the data, that is wanted. But this rigid concept has to underly any further concept.

The history of the cost of living problem has very largely been a search for an algebraical formula, involving expenditure data belonging just to the base and current occasion, which could with good reason be considered as an index of the cost-of-living. An index of such a form and with such a justification was never found. But one may ask: why should the index be given by an algebraical formula; why should it involve just the expenditure data for the base and object occasions; why, in view of the necessary indeterminacy arising from incomplete knowledge of preferences, should it be given a point-determination; and finally, why should it always be properly definable, seeing that it depends for its meaning on preferences, the existence of which requires a consistency of behaviour which may not be borne out by the data? The answer appears negative for all these questions.

5. Expenditure data.

A vector pair u, x which, being such that $u'x = 1$ represents a balance u together with a composition x which is on it, will be denoted by $[u;x]$. It is taken to define an expenditure figure

$$E = [u;x]$$

which, geometrically, can be viewed as an hyperplane - the locus of compositions on the balance u - together with a particular point on that hyperplane - the composition x. Thus, with any purchase (p, x), there is associated the expenditure figure $[u;x]$, where u is the associated balance.

Now, given an expenditure system f, to every u there corresponds an expenditure figure $E = [u;x]$ where $x = f(u)$, which is said to belong to f.

Any collection of expenditure figures defines an expenditure configuration. The collection $\mathcal{E}_f = \{E\}$ of expenditure figures E belonging to an expenditure system f defines the expenditure configuration associated with f; and any subconfiguration $\mathcal{E} \subset \mathcal{E}_f$ defines an expenditure subconfiguration of f. Given any expenditure configuration $\mathcal{E}$, it is called complete if it is of the form $\mathcal{E} = \mathcal{E}_f$, that is, if it is the configuration belonging to some system. A complete configuration defines a completion of every configuration containing it. Thus if $\mathcal{E}$, $\mathcal{F}$ are two configurations of which $\mathcal{E}$ is complete and such that $\mathcal{F} \subset \mathcal{E}$, then $\mathcal{E}$ is a completion of $\mathcal{F}$. Assuming balance and composition domains which are infinite, a finite configuration $\mathfrak{F}$ cannot be complete, but has a variety of completions.

Every expenditure configuration $\mathfrak{F}$ has a preference relation $P_{\mathfrak{F}}$ defined by

$$P_{\mathfrak{F}} = \vec{Q}_{\mathfrak{F}} ,$$

where

$$xQ_{\mathfrak{F}}\, y \equiv \bigvee_{u} [u;x] \in \mathfrak{F} \wedge y \neq x \wedge u'y \leq 1 .$$

Then, for an expenditure system f, with associated configuration ξ_f,

$$P_f = P_{\xi_f} .$$

Also, if $\mathfrak{F} \subset \xi$, then $P_{\mathfrak{F}} \subset P_{\xi}$.

The a priori form of a preference structure is derived by association with an expenditure system which admits a preference structure. This association is the requirement of general method. But beyond this form, all that can be taken as known about preference must be by observation.

The data of observation, of the kind to be admitted, are expenditures on a variety of goods on a variety of similar occasions, obtaining goods in certain amounts at certain prices.

Let there be some $k + 1$ occasions, for example yearly periods, indexed by $r = 0, 1, \ldots, k$. Let $M_r = (p_r, x_r)$ be the market purchase, observed in occasion r, of a certain composition in some n goods at certain prices, given by vectors x_r, p_r of order n. Then the assembled data forms the scheme

$$M = \{M_r\}_{r = 0,1,\ldots,k} .$$

Associated with the purchase M_r, there is the expenditure figure

$$E_r = [u_r; x_r] ,$$

where $u_r = \frac{p_r}{e_r}$, and $e_r = p_r{}'x_r$ is the total money expenditure in the purchase. These expenditure figures together form a finite expenditure configuration

$$\mathfrak{F} = \{E_r\}_{r=0,1,\ldots,k} .$$

The expenditure configuration $\mathfrak{F}$ is directly associated with the scheme of market data M :

$$M \longrightarrow \mathfrak{F} .$$

But the same configuration $\mathfrak{F}$ could be derived from different possible data M . While M has a direct observational meaning, it is taken as relevant to preference structure only in reduction to $\mathfrak{F}$; that is, the preference analysis of M is to be just an analysis of $\mathfrak{F}$.

It is true, there can be all kinds of other data: answers to questionnaires, the weather, and so forth. But it is taken, so as to limit and define our subject, that what is significant is not what an agent, such as a consumer, says - which may not be believed, and is in any case, generally of unfathomable significance - but what the agent does. Assuming the standard picture of the market, whatever the difficulties in that, what the consumer does is something in an observable framework, with coordinates which are, in principle, rather definite. As for the weather and such factors, of undoubted relevance to preference, it is not yet known how to profitably take them into account, and the first step is to ignore them. It is manifest in immediate experience that such factors are forces for change of preference. But an entity must be known in some proper sense before it can seriously be said to change. And the first task, before there can be any analysis of change of preference, is to have the framework in which the preferences are to be taken as known.

As for the additional data, which is contained in M but not in $\mathfrak{F}$, this is relevant to an analysis which exceeds preference analysis as here considered, which relates to the way a given expenditure is allocated to some different commodities when prices are given. It touches on demand analysis which takes that expenditure on the commodities not as independently given but as dependent on the prices. Such an analysis could be considered a preference analysis in a wider scheme, in which there is considered the additional choice possibilities of reserving money for other use, possibly spending it elsewhere, or of spending it directly on the considered commodities. The commodities are wanted goods, and therefore make a claim on expenditure; and this analysis would have interest in the strength of this claim in competition with other claims, which expresses demand.

Given a finite expenditure configuration $\mathfrak{Y}$, such as may be derived from market data obtained by observation, the general method is to consider all its completions; in particular all its completions limited by requirement of having a certain structure, such as continuity, regularity, differentiability, or of conforming to some algebraical model. When preference structure is the goal of analysis, there is added the requirement of preference consistency, in the exact, and also possibly in some approximate sense. The approximate sense would be such as to allow explanation of whatever is observed as belonging to a consistent system but with discrepancies due to error and disturbance. Here, however, there is to be confinement to the strict analysis; but this must be recognized as of over-limited scope, when, as is usual, there is a large and somewhat disordered accumulation of data to be taken into a single analysis.

6. Admissible hypotheses.

Let it be supposed that some market data M has been obtained, by observation on some consumer, in respect to some n commodities in some $k+1$ occasions, and that from this there has been obtained an expenditure configuration $\mathfrak{F}$, of $k+1$ figures in n dimension.

There is to be considered a totality of preference hypotheses for the consumer that is to be admitted on the data. This totality requires not the entire data M, but the configuration $\mathfrak{F}$ derived from it.

The general method is to suppose that observed acts which each single out a composition x_r on a certain balance u_r, are elements in a systematic behaviour which, in potentiality, singles out a composition $x = f(u)$ on every possible balance u according to some rule f. Thus, first there is considered the totality of expenditure systems f admitted by $\mathfrak{F}$, being such that $x_r = f(u_r)$ $(r = 0,1,\ldots,k)$.

But usually, in taking such a completion of observations, a structure is imposed on that completion, such as continuity or differentiability. This is the general method of forming expectations from observations, making use of a priori smoothness or other structural assumptions about behaviour. But here there is special concern with preference as an hypothesis of behaviour control. It happens that if f is proper, being associated with a preference system P, being of the form $f = f_P$, then this requires the continuity of f, and also certain differentiability properties. A certain smoothness is thus a built-in aspect of the preference hypothesis. Moreover, if f is of this form, and also regular, then P must be a scale, admitting a numerical function φ as a gauge. This coincides with the usual picture of consumer behaviour, in which there is supposition of a "utility" function regulating expenditures. If the function φ is differentiable, having gradient $g = g(x)$, then, since

$\varphi = \varphi(x)$ is to be a minimum under the constraint $u'x = 1$, there have to be the equilibrium conditions

$$g_r = u_r \lambda_r \quad (r = 0,1,\ldots,k) ,$$

where $g_r = g(x_r)$, and $\lambda_r = x_r' g_r$ since $u_r' x_r = 1$.

The concept is thus arrived at of the existence of a numerical function $\varphi = \varphi(x)$ which determines the composition x on any balance u as the maxima of φ under the constraint $u'x = 1$. For this determination to be always possible, φ has to be strictly increasing, and have strictly convex levels, in the domain of x. Let x be confined to a compact composition domain, C. Then it can be shown that φ can be chosen to be a convex function in that domain. Such a function can be smoothed into a continuously differentiable function, or even a function with any number of derivatives, which approximates it arbitrarily closely; so, empirically, there is no further limitation in considering only continuously differentiable preference functions, though the analysis while then somewhat simplified, need not be thus confined.

A proper preference system P such that $P_{\mathfrak{F}} \subset P$ is to be considered a preference hypothesis admissible on the data $\mathfrak{F}$. It can be shown that such exist if and only if $P_{\mathfrak{F}}$ is an order, by which condition the configuration $\mathfrak{F}$ may be said to be consistent.

Rather than consider all proper preference systems which are admissible on $\mathfrak{F}$, there may be considered the class $\mathcal{S}_{\mathfrak{F}}$ among these which are normal preference scales, it being known that these at least include the admissible regular preference scales, that is all those proper preference systems which are regular. For these preference systems $\mathcal{S}_{\mathfrak{F}}$, it is known that they are represented in some convex neighbourhood of the points x_r by a numerical function φ which is strictly increasing and, beyond having strictly convex levels, is strictly convex. The same scale S corresponds to a variety of functions. But to each function there corresponds just one scale measured by it. The totality $\mathcal{S}_{\mathfrak{F}}$ of hypotheses admissible on $\mathfrak{F}$ can be approached by way of the totality $\Gamma_{\mathfrak{F}}$ of such convex functions φ which represent them.

A method of preference analysis consists in analysis of the totality $\mathcal{S}_{\mathfrak{F}}$ of preference hypotheses admissible on an empirical configuration $\mathfrak{F}$, and investigation of the range of any characteristic of a scale S as S ranges in $\mathcal{S}_{\mathfrak{F}}$. It is a rather rigid method, since $\mathfrak{F}$ may be disorderly, in which case $\mathcal{S}_{\mathfrak{F}}$ will be empty. One could do better in some respects by generally viewing $\mathfrak{F}$ as disturbed from an orderly configuration on some constructive model. But this method nevertheless is the first and most fundamental one. Mathematically it depends on investigation of the class $\Gamma_{\mathfrak{F}}$ of function φ which are strictly increasing and convex and under the constraints $u_r'x = 1$ $(r = 0,1,\ldots,k)$ have maxima at $x = x_r$.

7. Ranging the cost of living.

The index-number problem of the cost of living has always been a conceptual problem: how to turn the somewhat vague question as to the cost of maintaining a given standard of living into a definite principle of measurement. There need not be just one way of doing this, because the question thus simply put is loose, and there are some different senses for its intelligibility. But there is a most direct sense which can be singled out, which has first claim for consideration, though its investigation is far from marking the whole problem. A clarification has to be made of the meaning of having preferences. Economic theory commonly proceeds on the assumption that all agents have preferences regulating their choices. But consumers do not know their preference systems. And economists do not know them either, though they have talked about them in the abstract. And it has to be recognized that to make choices is not necessarily to have preferences. So it may well be asked what can be the significance of such a theory.

However, the hypothesis may be admissible on given data; and then it is possible to proceed directly, and consider all feasible hypotheses.

Given an expenditure configuration $\mathfrak{F}$, which presents the data of observation, there is a class $\mathcal{S}_{\mathfrak{F}}$ of preference systems which are admissible hypotheses, and, if there are any, each, without any further principle of limitation, is equally a candidate for construction as an hypothesis. Relative to any one system $S \in \mathcal{S}_{\mathfrak{F}}$, there is a determination $\rho_{rs}(S)$ of the multiplier of expenditure in occasion r so as to maintain, at the prices of occasion r, a standard of living equivalent to that found in occasion s. It is asked: what is the range of $\rho_{rs}(S)$ as S ranges in $\mathcal{S}_{\mathfrak{F}}$? This is a definite question, and it has a definite answer: the range is an interval, whose lower and upper limits $\rho^{i}_{rs}(\mathfrak{F})$, $\rho^{o}_{rs}(\mathfrak{F})$ can be determined from $\mathfrak{F}$ by certain formulae. These formulae will now be described. But it has to be admitted that, though in them a principle is made perfectly precise, it is not obvious how to evaluate them.

The <u>cross-structure</u> of the expenditure configuration $\mathfrak{F} = \{E_r\}$ with figures $E_r = [u_r; x_r]$ is defined by the array $D_{\mathfrak{F}} = \{D_{rs}\}$ of <u>cross-deviations</u> $D_{rs} = u_r' x_s - 1$ between its figures. $(r \neq s;\ r,s = 0,1,\ldots,k$ The following conditions in the cross-structure are equivalent.

I. $D_{rs} \leq 0$, $D_{st} \leq 0$, ..., $D_{qr} \leq 0$
impossible for distinct $r,s,t,\ldots,q$ taken from $0,1,\ldots,k$.

II. $\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qr} > 0$

for some $\lambda_r > 0$ and for all distinct $r,s,t,\ldots,q$ taken from $0,1,\ldots,k$.

III. $\lambda_r > 0$, $\lambda_r D_{rs} > \varphi_s - \varphi_r$ $(r \neq s;\ r,s = 0,1,\ldots,k)$

for some (λ_r, φ_r) .

Let $\Lambda = \{\lambda_r\}$ and $\Phi = \{\varphi_r\}$. Then, moreover, Λ satisfies II. if and only if (Λ, Φ) satisfies III. for some Φ .

Condition I., which is the Houthakker condition applied finitely, to the finite configuration $\mathfrak{F}$, is the condition that $P_{\mathfrak{F}}$, which is the minimal transitive relation with the property

$$D_{rs} \leq 0 \Longrightarrow x_r P \ x_s \ ,$$

be irreflexive, and therefore an order. It is a necessary condition that $\mathfrak{F}$ admit a normal preference hypothesis, that is $\mathcal{S}_{\mathfrak{F}} \neq 0$; and it can be shown also to be a sufficient condition.

Thus all these three conditions are equivalent to the condition $\mathcal{S}_{\mathfrak{F}} \neq 0$ for the existence of a normal preference scale S which is an admissible hypothesis for $\mathfrak{F}$. In this and only this case, Λ which satisfies II. and (Λ, Φ) which satisfy III., exist.

Let (Λ, Φ) now denote any solution of III., so that Λ denotes any solution of II. Then Λ, Φ define <u>level and multiplier sets</u> for $\mathfrak{F}$. Let $\alpha = \{\alpha_r\}$ be any set of numbers $\alpha_r \geq 0$ such that $\Sigma\alpha_r = 1$, and let

$$x_\alpha = \Sigma x_r \alpha_r \ , \ \varphi_\alpha = \Sigma \varphi_r \alpha_r \ .$$

Then

$$\rho^{i}_{rs}(\mathfrak{F}) = \min_{\Lambda,\Phi} \min_{x} \{u_r{}'x;\ (x - x_t)' u_t \lambda_t \geq \varphi_s - \varphi_t (t = 0,1,\ldots,k)\}$$

and

$$\rho^{o}_{rs}(\mathfrak{F}) = \max_{\Lambda,\Phi} \min_{\alpha} \{u_r{}'x_\alpha \ ;\ \varphi_\alpha \geq \varphi_s\} \ .$$

<u>For any</u> ρ_{rs} <u>there exists a scale</u> $S \in \mathcal{S}_{\mathfrak{F}}$ <u>such that</u> $\rho_{rs}(S) = \rho_{rs}$ <u>if and only if</u>

$$\rho^{1}_{rs}(\mathfrak{F}) < \rho_{rs} < \rho^{0}_{rs}(\mathfrak{F}) \ .$$

Let this range for the cost-of-living index ρ_{rs} be called the <u>absolute range</u>, on the data provided by the expenditure configuration .

Some of the preference systems in $\mathcal{S}_{\mathfrak{F}}$ may be rather pathological, by which we may take to mean having levels which are not well-rounded. We may attempt to be more selective, and through this obtain a narrower range for ρ_{rs} corresponding to more agreeable systems in $\mathcal{S}_{\mathfrak{F}}$. Instead of asking of level and multiplier sets Λ, Φ that

$$\lambda_r D_{rs} > \varphi_s - \varphi_r > - \lambda_s D_{sr} \ ,$$

in other words that the differences $\varphi_s - \varphi_r$ lie in the intervals $[-\lambda_s D_{sr}, \lambda_r D_{rs}]$, one may ask more restrictively that they lie at the mid-points of these intervals. That is,

$$\varphi_s - \varphi_r = \frac{1}{2}(\lambda_r D_{rs} - \lambda_s D_{sr}) \ .$$

Such a Λ, Φ will be called a <u>median solution</u>. Of course, a median solution may not exist. But if it does, with $\lambda_r > 0$, in which case <u>median consistency</u> may be said to hold for $\mathfrak{F}$, the range of ρ_{rs} may be narrowed to correspond to it, defining a <u>median range.</u> A median solution always exists for $k \leq 3$, but need not have $\lambda_r > 0$. If $k = 3$, an essentially unique median solution always exists. So for $k = 3$, we only have to ask if $\lambda_r > 0$. The concept of a median solution may be given a general scope by suitably defining a median discrepancy for any level and multiplier sets, and seeking those with minimum discrepancy.

8. Algebraical method.

The consistency of $\mathfrak{J}$ gives definition of an absolute range for ρ_{rs}, which is hard to calculate, and the more restrictive median consistency, if it holds, provides the narrower median range, which is somewhat easier to calculate. The calculations are not made according to an explicit algebraical formula, but are of a combinatorial type, involving solutions of systems of linear inequalities. It would be of advantage if an algebraical formula could be found, locating a range within the absolute or even the median range, and especially so if this corresponded to some peculiarly interesting class of hypotheses. Such a class is provided by preference scales which can be measured in a convex neighbourhood of the points x_r by a quadratic function. If any such are admissible, then will be said to satisfy the condition of quadratic consistency. Such a quadratic hypothesis can perhaps be taken as the model of the smoothest, least pathological, of the admissible hypotheses. The ranging of the cost of living with respect to this class could be significant in this respect. Also the quadratic hypothesis is a constructive algebraical model which is the natural model for a statistical analysis, in which it has a role analogous to that of the linear hypothesis in regression analysis.

The algorithm for ranging a cost of living measurement on the quadratic hypothesis applies to a configuration of four figures $(k = 3)$. This is connected with the fact that only in this case of four figures a median solution generally exists and is essentially unique; and also it is bound up with the peculiar relation between median consistency and quadratic consistency. Given quadratic consistency, the quadratic range for ρ_{rs} will be determined as a subinterval of the absolute range. Thus there are three consistency conditions of increasing strength and correspondingly, three intervals for ranging the cost of living, one lying inside another. But the stronger the condition required, the narrower the scope of the associated method, so the calculation on the quadratic method has least scope of the three. Nevertheless, it has special importance, for the reasons indicated, and since any configuration can be regarded as obtained by disturbance of a quadratically consistent configuration. This, however, is not the concern here, where discussion is to be confined to an exact rather than an approximate kind of analysis.

ALGORITHM. Let $\mathfrak{F} = \{E_r\}$ be an expenditure configuration of four figures $E_r = [u_r;x_r]$ $(r = 0,1,2,3)$, with cross deviations determined by

$$D_{rs} = u_r{}'x_s - 1 \ .$$

Let $\Lambda = \{\lambda_r\}$ be four multipliers whose three independent ratios are determined from the cycle-reversibility equations

$$C_{o12} = 0 \ , \ C_{o23} = 0 \ , \ C_{o31} = 0$$

where

$$C_{ors} = (\lambda_o D_{or} + \lambda_r D_{rs} + \lambda_s D_{so}) - (\lambda_s D_{sr} + \lambda_r D_{ro} + \lambda_o D_{os}) \ .$$

Then let $\Phi = \{\varphi_r\}$ be four levels whose intervals are determined from the median equation

$$\varphi_r - \varphi_o = \tfrac{1}{2}(\lambda_o D_{or} - \lambda_r D_{ro}) \ .$$

These multipliers and levels are uniquely determined by arbitrarily taking

$$\lambda_o = 1 \ , \ \varphi_o = 0 \ .$$

Now let

$$g_r = u_r \lambda_r$$

and form the matrices

$$X_o = \{x_r - x_o\}\ ,\ G_o = \{g_r - g_o\}$$

of order $n \times 3$. The 3×3 matrix $X_o'G_o$ should be symmetric, by consequence of the conditions determining the numbers λ_r.

The criterion for quadratic consistency is $\lambda_1, \lambda_2, \lambda_3 > 0$, and that $X_o'G_o$ be negative definite.

Calculate

$$\tilde{\tilde{\delta}}_{rs} = \frac{\varphi_s - \varphi_r}{\lambda_r} .$$

Now calculate

$$\hat{M} = \varphi_o + \tfrac{1}{2} g_o' X_o (X_o' G_o)^{-1} X_o' g_o .$$

Quadratic consistency provided, it is necessary that

$$\varphi_r < \hat{M} ,$$

so it is possible to calculate

$$\hat{X}_r = -\{-2(\varphi_r - \hat{M})\}^{\frac{1}{2}} ,$$

$$\hat{U}_r = -\frac{\hat{X}_r}{\lambda_r} ,$$

and then

$$\hat{\delta}_{rs} = U_r(X_s - X_r) .$$

With quadratic consistency given, it should be found automatically that

$$\tilde{\delta}_{rs} < \hat{\delta}_{rs} < D_{rs} .$$

Assuming $n > 4$, quadratic consistency implies the existence of an infinity of normal completions of the configurations which belong to quadratic preference functions. Any one of these gives a determination of the fractional change δ_{rs} in the expenditure of occasion r which exactly compensates for the price-change from occasion s. The totality of these determinations describes the open interval

$$\tilde{\delta}_{rs} < \delta_{rs} < \hat{\delta}_{rs} .$$

9. Computer program.

This program for carrying out the cost-of-living algebraical algorithm is written in the Bell II language for the IBM 650 computer. All standard console settings for Bell II should be used.

INPUT:

The input consists of the program deck, a control-card, and the price and quantity data. The control-card stores +0 000 000 N and +0 000 N 000 in addresses 299 and 300, respectively. If there are 36 commodities in our example, addresses 299 and 300 will read +0 000 000 036 and +0 000 036 000; if two commodities, they will read +0 000 000 002 and +0 000 002 000. The maximum number of commodities that this program can accommodate is 50. The price vectors are put in consecutive addresses beginning at 301, 351, 401, and 451, while the quantity vectors start at 501, 551, 601, and 651.

For example, when there are 30 commodities, the control-card would read:

in columns 7-9	299
in column 10	2
in columns 11-21	+0 000 000 030
in columns 22-32	+0 000 030 000

The price and quantity vectors will be located as follows:

P_0	in addresses 301-330
P_1	in addresses 351-380
P_2	in addresses 401-430
P_3	in addresses 451-480
x_0	in addresses 501-530
x_1	in addresses 551-580
x_2	in addresses 601-630
x_3	in addresses 651-680

The input should have the following order:

1. The Bell II deck
2. The problem number card
3. The program deck
4. A "zero card"
5. For each standard problem
    a. The control-card
    b. All price and quantity data

OUTPUT:

The output will have the following form:

1	701	
2	705	4 x 4 D_{rs} row-wise matrix
3	709	
4	713	
5	732	1 x 4 λ vector
6	832	1 x 4 φ vector
7	720	
8	723	3 x 3 $G_o{}'X_o$ row-wise matrix
9	726	
10	719	value of $G_o{}'X_o$ determinant
	901	901 if $G_o{}'X_o$ is negative definite
11	or	
	902	902 if $G_o{}'X_o$ is not negative definite
12	836	value of $\hat{M}$
13	751	
14	755	4 x 4 $\hat{\delta}_{rs}$ row-wise matrix if all $\hat{M} > \varphi_r$
15	759	
16	763	
17	776	
18	780	4 x 4 $\tilde{\delta}_{rs}$ row-wise matrix
19	784	
20	788	

The first column, which is the number of the output card, has no significant meaning for our problem. The second column lists the addresses of the first number in the given row. When we are calculating more than one standard problem, this column will be repeated for each standard problem.

Column / Card No.	Loc. 7-9	N 10	Field 1 11-21	Field 2 22-32	Field 3 33-43	Field 4 44-54	Field 5 55-65	Field 6 66-76
01	001	6	+9800049000	+7000299300	+6006299006	+9800047000	+9100011000	+7000301300
02	007	6	+8150008004	-6006299006	+9800048000	+1900900832	+9800001000	+9100110000
03	013	6	+9200110000	+9300001000	+4301501801	+0200300000	+8101000011	+9800002000
04	019	6	+9100101000	+9200101000	+9300010000	+3301801301	+0200300000	+8101000018
05	025	6	+8250004026	+9800003000	+8301004009	+9800004000	+1900900899	+9800005000
06	031	6	+9100110000	+9200010000	+9300001000	+9400100000	+9500001000	+4301501701
07	037	6	+0200300000	+8101000030	+9300101000	+9500101000	-1701901701	+8250004043
08	043	6	+9800028000	+8301004028	+8450004046	+9800029000	+8504004028	+7300701704
09	049	6	+7300705708	+7300709712	+7300713716	-1707708720	-1712710721	-1714715722
10	055	6	+7202899723	-1709712725	-1715713726	-1703704727	-1708705728	+1900900729
11	061	6	-1713714730	-1704702731	+3720728732	+9800006000	+9100101000	+9200010000
12	067	6	+2728732728	+9100111000	+9200100000	-1720728728	+8101004064	+9200000010
13	073	6	+3725729736	+8204002064	+7201901732	+3731730735	-5726727000	-3000725734
14	079	6	-5709703738	-3738902834	+2722735000	+4721734722	-3722720733	-5705702737
15	085	6	-3737902833	+2713735739	-1739704739	-3739902835	+7300732735	+7300832835
16	091	6	+9800009000	+9100101000	+9200101000	+9300010000	+2351733351	+9100111000
17	097	6	+9200101000	-1351301351	+9100111000	+9200101000	-1551501551	+0200300000
18	103	6	+8101000091	+8250003105	+9800011000	+8301003091	+9800012000	+1900900736
19	109	6	+9800013000	+9100110000	+9200010000	+9300001000	+9400100000	+9500001000
20	115	6	+4351551720	+0200300000	+8101000109	+8250003119	+9800030000	+8301003107
21	121	6	+8450003122	+9800031000	+8503003107	+7300720722	+7300723725	+7300726728
22	127	6	+2720724000	-4721723717	+2721726000	-4720727716	+2721725000	-4722724715
23	133	6	+2716901714	+2720728000	-4722726713	+2722727000	-4721728712	+2715901711
24	139	6	+2712901710	+2724728000	-4725727709	-2717901718	+2722711000	+4725714000
25	145	6	+4728717719	+7300719719	+8700718152	+8700719152	+8700720152	+7300901901
26	151	6	+8000000154	+9800014000	+7300902902	+9800015000	+9100101000	+3709719709
27	157	6	+8101009154	+9800016000	+1900900000	+9800017000	+9100110000	+9300010000
28	163	6	+9400001000	+4301551720	+0200300000	+8101000160	+9800018000	+9100011000
29	169	6	+9400100000	+9500011000	+2720709709	+9200011000	+9400111000	+2720709709
30	175	6	+8101003176	+9800019000	+8203003167	+8350003179	+9800020000	+8401003181
31	181	6	+9800027000	+8503003158	+9800021000	+9100111000	+1709710710	+8101008183
32	187	6	-3717902836	+7300836836	+9800022000	+9100010000	-1717736000	+8700000217
33	193	6	-2901000000	+9100001000	+0300000740	+9100111000	+3740732744	+8101004189
34	199	6	+9800023000	+9100010000	+9200100000	-1740740000	+9100001000	+9200100000
35	205	6	+9300001000	+5744900776	+8101004199	+8201004209	+9800024000	+8304004199
36	211	6	+8600899227	+7216776751	+7300751754	+7300755758	+7300759762	+7300763766
37	217	6	+9800041000	+9600001000	+9800025000	+9100011000	+3901732744	+9100101000
38	223	6	+3736902740	+8101004219	+1899901899	+8000000199	+9800026000	+7300776779
39	229	4	+7300780783	+7300784787	+7300788791	+8000000001		

PROGRAM

EXAMPLE

In the following example we show the price and quantity data (Table 1), how it should be punched on the input cards (Table 2), the output from the computer (Table 3), and the translated output (Table 4).

TABLE 1

PRICE AND QUANTITY DATA

1A. Price Data

	p	Year 0	1	2	3
Commodity	1	5.0636	5.2906	5.6218	6.5089
	2	1.27	2.07	3.58	2.40
	3	3.03	4.78	10.66	7.00
	4	.1233	.1217	.1242	.1695

1B. Quantity Data

	x	Year 0	1	2	3
Commodity	1	6.90	6.75	7.83	7.69
	2	477.	502.	552.	583.
	3	46.3	54.5	56.7	55.0
	4	10,400.	12,640.	14,880.	15,680.

TABLE 2

PRICE AND QUANTITY INPUT CARD FORMAT

Card No.	Loc. (Columns 7-9)	N (Columns 10)	Field 1 (Columns 11-21)	Field 2 (Columns 22-32)	Field 3 (Columns 33-43)	Field 4 (Columns 44-54)
1	299	2	+0000000004	+0000004000		
2	301	4	+5063600050	+1270000050	+3030000050	+1233000049
3	351	4	+5290600050	+2070000050	+4780000050	+1217000049
4	401	4	+5621800050	+3580000050	+1066000051	+1242000049
5	451	4	+6508900050	+2400000050	+7000000050	+1695000049
6	501	4	+6900000050	+4770000052	+4630000051	+1040000054
7	551	4	+6750000050	+5020000052	+5450000051	+1264000054
8	601	4	+7830000052	+5520000052	+5670000051	+1488000054
9	651	4	+7690000050	+5830000052	+5500000051	+1568000054

TABLE 3

OUTPUT CARD FORMAT

Loc. (Columns 7-9)	N (Columns 10)	Field 1 (Columns 11-21)	Field 2 (Columns 22-32)	Field 3 (Columns 33-43)	Field 4 (Columns 44-54)
701			+1609182049	+3314315049	+3954783049
705		-1262368049		+1365295049	+1896547049
709		-2103895049	-1088228049		+4279990048
713		-2705695049	-1601283049	-4389760048	
732		+1000000050	+1149631250	+1295415350	+1164408250
832			+1530219849	+3019866449	+3552658249
720		-1579233448	-2934717248	-3231949948	
723		-2934716848	-5888978048	-6749290048	
726		-3231940948	-6749286848	-8042486248	
719		-1092431043			
901		+1000000050			
836		+9772975049			
751			+1595324649	+3298124249	+3952232849
755		-1274422149		+1360278449	+1882811949
759		-2116393749	-1092680049		+4197390048
763		-2707884549	-1614843049	-4481645448	
776			+1530219849	+3019866449	+3552658249
780		-1331052849		+1295760449	+1759206349
784		-2331195649	-1149937549		+4112903448
788		-3051041949	-1736880949	-4575644548	

TABLE 4

THE TRANSLATED OUTPUT

(rounded to 4 decimal places)

4A. D - Matrix

	D	Year 0	1	2	3
Year	0		.1609	.3314	.3955
	1	-.1262		.1365	.1897
	2	-.2104	-.1088		.0428
	3	-.2706	-.1601	-.0439	

4B. λ and φ - Vectors

	Year 0	1	2	3
λ	1.0	1.1496	1.2954	1.1644
φ	0.0	.1530	.3020	.3553

4C. $G_o'X_o$ Matrix

	$G_o'X_o$	Year 1	2	3
Year	1	-.0158	-.0293	-.0323
	2	-.0293	-.0589	-.0675
	3	-.0323	-.0675	-.0804

$$|G_o'X_o| = -1.092 \cdot 10^{-7}$$

$G_o'X_o$ is negative definite

$\hat{M} = .9773$

TABLE 4 (cont.)

4D. $\hat{\delta}$ - Matrix

	$\hat{\delta}$	Year 0	1	2	3
Year	0		.1595	.3298	.3952
	1	-.1274		.1360	.1883
	2	-.2116	-.1093		.0420
	3	-.2708	-.1615	-.0448	

4E. $\tilde{\delta}$ - Matrix

	$\tilde{\delta}$	Year 0	1	2	3
Year	0		.1530	.3020	.3553
	1	-.1331		.1296	.1759
	2	-.2331	-.1150		.0411
	3	-.3051	-.1737	-.0458	

10. Illustrations.

A number of numerical examples will now be shown, to illustrate the algebraical method of ranging the cost of living. The calculations have been carried out on the IBM 650 computer, using the program which has just been given.

Example 1. In the construction of this example, there is first taken, somewhat arbitrarily, a price and a quantity vector

$$p_o = (2,3,4) \ , \ x_o = (5,6,7) \ ,$$

making a total expenditure of

$$p_o{}'x_o = 2 \times 5 + 3 \times 6 + 4 \times 7 = 56$$

in an initial year, which is to be kept fixed as the prices subsequently change. Now each of the prices is raised one unit, in turn. When one price is raised a unit, the corresponding quantity is diminished a unit, and the other quantities raised a unit. The resulting quantities are then scaled proportionately, so as to bring the expenditure to the fixed total.

All that is intended in this procedure is to construct artificially a scheme of data that conforms to the normal idea that when a commodity rises in price, while other prices remain fixed, less of this commodity is bought, and, instead, there is an increase in the quantities of other commodities, whose prices have remained the same.

In this way, there is obtained the following example, involving three goods 1,2,3 and four years 0,1,2,3.

p	0	1	2	3
1	2	3	2	2
2	3	3	4	3
3	4	4	4	5

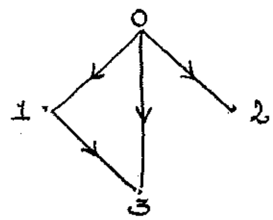
x	0	1	2	3
1	5	3.446	5.250	5.333
2	6	6,031	4.375	6.222
3	7	6.892	7.000	5.333

The cross structure is given by the array of cross-deviation

$$D_{rs} = \frac{p_r{}' x_s}{p_r{}' x_r} - 1 :$$

D	0	1	2	3
0	0	-.0615	-.0781	-.0952
1	.0893	0	.0156	(0)
2	.1071	.0462	0	.0159
3	.1250	.0615	.0469	0

From this is constructed the graph of the preference relations.



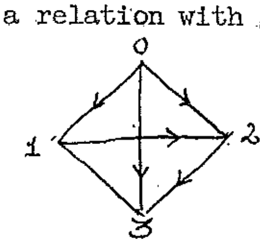
The preferences are thus consistent, and the relation is an order. It is a partial order, since year 2, determined in relation to year 0, is undetermined in relation to years 1 and 3.

The multipliers and levels are

	λ	φ
0	1	0
1	1.416	- .0940
2	1.438	- .1161
3	1.461	- .1389

The multipliers are all positive.

The level-order of the years, following the φ-magnitudes, is thus 0, 1, 2, 3 . This is a relation with graph



It is thus a complete order, which is a refinement of the partial order first obtained.

The matrix $G_o{}'X_o$ is

$G_o{}'X_o$	1	2	3
1	-.0649	-.0262	-.0312
2	-.0262	-.0759	-.0360
3	-.0312	-.0360	-.0874

This is negative definite.

Since the multipliers λ are positive and the matrix $G_o'X_o$ is negative definite, the condition of quadratic consistency is satisfied.

The peculiar method of constructing the example was followed in the hope that it would provide quadratically consistent data, and here that hope is fulfilled. This does not mean the hope would always be fulfilled: that is a plausible conjecture, which does not happen to be true, as can be shown by a counter-example. All the same, it would be interesting to have general rules for artificially constructing "good" examples.

Accordingly, to the question, "Do there exist normal preference systems which, on the data, are admissible hypotheses, and which, in a convex neighbourhood of the points x_r ($r = 0,1,2,3$) , are represented by quadratic functions?" the answer now is "Yes."

It may be added, further to this answer, that there is an infinity of such quadratic functions; however, rather remarkably, when they are each normalized by addition and multiplication with suitable constants, they all take the same values φ_r at the points x_r ($r = 0,1,2,3$). Thus the order determined for the years relative to each admissible quadratic preference hypothesis is invariant throughout the entire infinite class, and identical with the φ-order determined by the algorithm. This φ-order is a refinement of the preference order P determined directly from the cross-structure. While P is often a partial order, and can even be null, this φ-order is almost always a complete order.

It may now be asked what is the range of a cost of living measurement δ_{rs} corresponding to all admissible quadratic preference hypotheses. In view of quadratic consistency, admissible quadratic hypotheses do exist for this example; so this range is defined for all $r,s = 0,1,2,3$. The answer is an interval whose lower and upper limits $\tilde{\delta}_{rs}$, $\hat{\delta}_{rs}$ according to the algorithm, are:

$\hat{\delta}$	0	1	2	3
0	0	-.0739	-.0877	-.1012
1	.0806	0	-.0151	-.0298
2	.1004	.0159	0	-.0150
3	.1209	.0327	.0161	0

$\tilde{\delta}$	0	1	2	3
0	0	-.0940	-.1161	-.1389
1	.0664	0	-.0156	-.0317
2	.0807	.0154	0	-.0159
3	.0951	.0308	.0156	0

It is verified in every case

$$\tilde{\delta}_{rs} < \hat{\delta} < D_{rs} .$$

Since $D_{rs} = \frac{e_{rs}}{e_{rr}} - 1$, where $e_{rs} = p_r{}'x_s$ is the cost of purchases in year s evaluated at the prices in year r, so that

$$e_{rs} = e_{rr} + e_{rr}D_{rs} ,$$

D_{rs} is the fractional increment of the expenditure e_{rr} in year r needed to raise it to the expenditure e_{rs} needed to purchase in year r which exactly compensates for the change of prices from year s to year r. Hence $\epsilon_{rs} = e_{rr} + e_{rr}\delta_{rs}$ is a determination of the cost, at prices of year r, of maintaining a standard of living equivalent to that in year s, and is a determination of the cost of living with years s and r as base and current periods relative to a preference hypothesis that is admissible on the data. And it appears, for any such δ_{rs}, that

$$\delta_{rs} < D_{rs} .$$

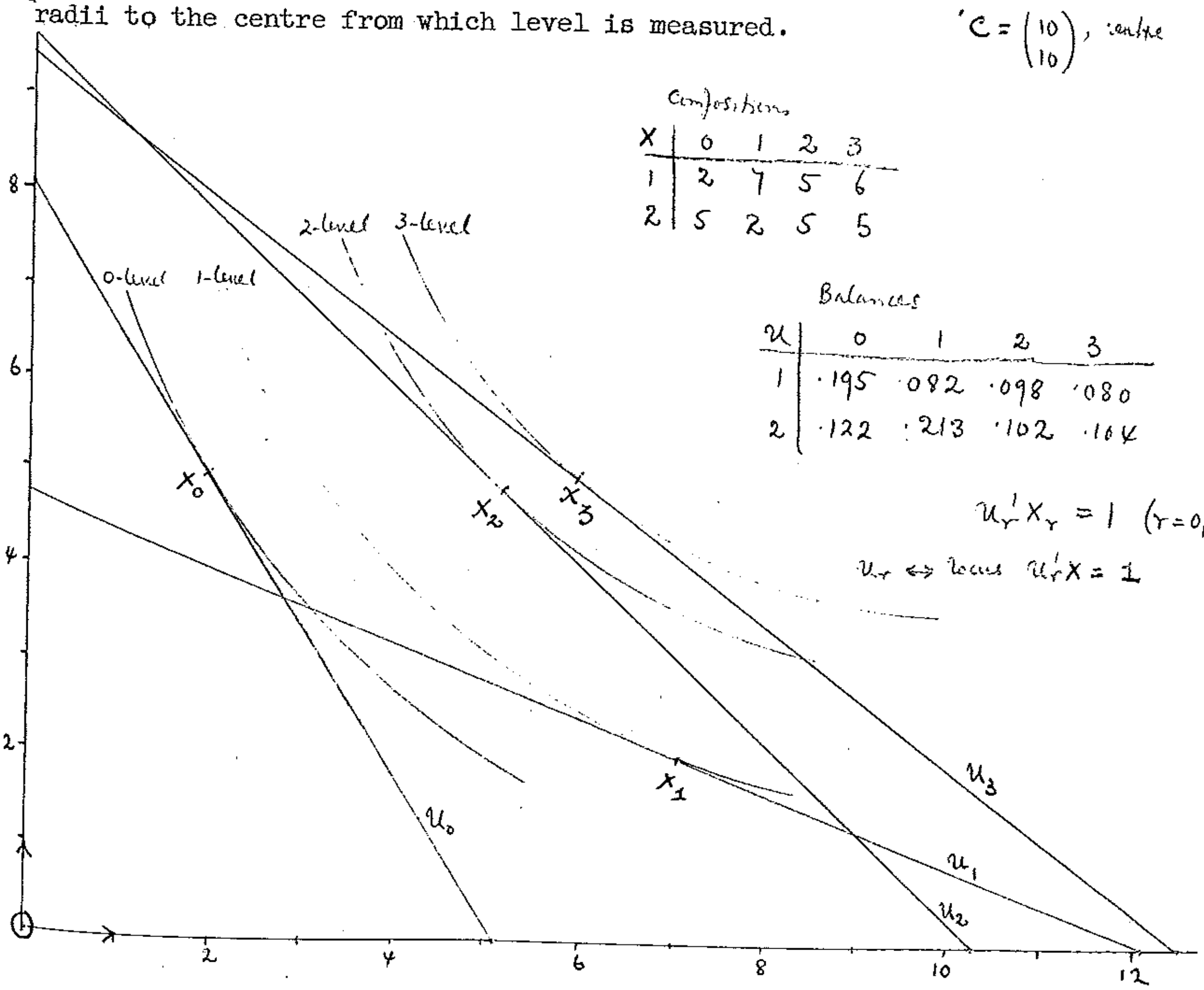
The numbers $\tilde{\delta}_{rs}$, $\hat{\delta}_{rs}$ calculated by the algorithm thus range the cost of living measurement δ_{rs} relative to a certain class of admissible hypotheses, and they bound it definitely less than the number D_{rs} which represents the conventional measurement.

<u>Example 2.</u> In constructing examples, instead of producing them first, and then making a trial to see if the admissible hypotheses exist, it is possible to start with a normal hypothesis on the quadratic model and produce examples directly from it, which will automatically have this desired consistency.

This will be done in two dimensions, for the advantage that graphical representation and direct geometrical measurement will then be possible.

For a particularly simple procedure, let preference level be measured in some suitable region by the distance squared from some suitable point. Take four points in the region, and the directions to the radii to the centre from which level is measured.

$C = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$, centre



Compositions

X	0	1	2	3
1	2	7	5	6
2	5	2	5	5

Balances

u	0	1	2	3
1	·195	·082	·098	·080
2	·122	·213	·102	·104

$u_r' X_r = 1 \quad (r = 0, 1, 2, 3)$

$u_r \leftrightarrow$ locus $u_r' X = 1$

The point-coordinates provide quantities, and these directions determine ratios of prices, since they are the directions of the gradient of level at those points.

Then, with $\hat{c} = (10, 10)$ as the centre of measurement, and

$$\varphi = -\{(x_1-10)^2 + (x_2-10)^2\}$$

and the points

x	0	1	2	3
1	2	7	5	5
2	5	2	5	6

there can correspond any prices parallel to the gradients

g	0	1	2	3
1	8	3	5	4
2	5	8	5	5

The price vectors normalized to make unit total expenditure are the uniquely determined balance vectors

u	0	1	2	3
1	.195	.082	.098	.080
2	.122	.213	.102	.104

The directly calculated multipliers $\lambda = x'g$ and levels φ are

	λ	φ
0	41	-44.5
1	37	-36.5
2	50	-25
3	49	-20.5

which are such that $g_r = u_r\lambda_r$, and these, normalized to make

$\lambda_o = 1$, $\varphi_o = 0$, are

	λ	φ
0	1	0
1	.902	.195
2	1.220	.476
3	1.195	.585

This normalization replaces φ by the function

$$\frac{\varphi + 44.5}{41}$$

with maximum

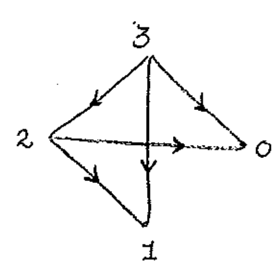
$$\hat{M} = \frac{44.5}{41} = 1.085$$

at the centre $\hat{c} = (10, 10)$.

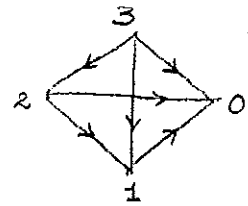
The algorithm, proceeding indirectly on just the cross-structure, will be found to reproduce these (λ, φ)-values exactly, the cross-structure being:

D	0	1	2	3
0	0	.610	.585	.780
1	.243	0	.486	.568
2	-.300	-.100	0	.100
3	-.327	-.224	-.082	0

The order determined just from cross-structure is the partial order P with graph



and the φ-order is the complete order 0, 1, 2, 3 with graph



which is seen to be a refinement of P .

Either by arithmetical calculation or by graphical measurement, there is obtained the table

δ	0	1	2	3
0	0	.205	.543	.697
1	-.206	0	.340	.494
2	-.334	-.208	0	.094
3	-.396	-.280	-.087	0

for the compensation number δ_{rs} based on the preference system with level measure by distance to the point $\hat{c}$.

The algorithm cannot apply direction, since here $n = 2$, and it generally requires $n \geq 3$. With $n = 2$, the matrix $G_o{}'X_o$ must be singular. Here it is

$G_o{}'X_o$	1	2	3
1	-.829	-.366	-.488
2	-.366	-.220	-.293
3	-.488	-.293	-.390

But for rounding error, it is found that $G_o{}'X_o$ is non-positive definite, and that its determinant vanishes. With the multipliers λ all positive, the criterion for quadratic consistency is satisfied, in the form it takes for two dimensions.

The following procedure will not be explained here, but it has its reason in the theory of calculation which underlies the algorithm. Delete the last row and column from $G_o{}'X_o$, to make

$$G_o{}'X_o = \begin{pmatrix} -.829 & -.365 \\ -.365 & -.219 \end{pmatrix} \qquad (G_o{}'X_o)^{-1} = \begin{pmatrix} -4.555 & 7.593 \\ 7.593 & -17.210 \end{pmatrix} .$$

Then

$$\begin{aligned} \hat{c} &= x_o - X_o(G_o{}'X_o)^{-1}X_o{}'g_o \\ &= \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} -4.555 & 7.593 \\ 7.593 & -17.210 \end{pmatrix} \begin{pmatrix} .609 \\ .585 \end{pmatrix} \\ &= \begin{pmatrix} 9.994 \\ 10.003 \end{pmatrix} \qquad \left(\text{approximately } \begin{pmatrix} 10 \\ 10 \end{pmatrix} \;!\right) . \end{aligned}$$

Also

$$\begin{aligned} \hat{M} &= -\tfrac{1}{2}(g_o{}'X_o)(G_o{}'X_o)^{-1}(x_o{}'g_o) \\ &= -\tfrac{1}{2}(.609 \quad .585) \begin{pmatrix} -4.555 & 7.593 \\ 7.593 & -17.210 \end{pmatrix} \begin{pmatrix} .609 \\ .585 \end{pmatrix} \\ &= 1.084 \quad (\text{approximately } 1.085 \;!) \end{aligned}$$

What has been done here is to reconstruct the centre $\hat{c}$ and the maximum value $\hat{M}$.

So far the following has been done: We had a circular preference system and derived some data from it, associated with four points. We had a function φ, representing the preference system, and calculated the multipliers and levels λ and φ at the four points, the function being normalized to make $\lambda_o = 1$, $\varphi_o = 0$. This function had maximum value $\hat{M} = 1.085$ at the centre $\hat{c} = \{10, 10\}$. Also, from the preference system, arithmetically or by geometrical measurement from the graph, the wanted numbers δ_{rs} are determined.

Now we take the data derived from the preference system (and, so to speak, forget how we got it). We verify quadratic consistency, so we know it can be derived from some preference system on the quadratic model, but we do not know the system (since we have forgotten it). Since $n = 2$, there can generally be at most one such system. From the data, we reconstruct the λ and φ values we had before, and also the centre $\hat{c}$ and maximum value $\hat{M}$ of some quadratic preference function, about which we know nothing further, beyond that it exists (not remembering it is the distance-squared from $\hat{c}$). The centre $\hat{c}$ was reconstructed just out of curiosity, to show how the calculation works. Our serious interest is in the numbers δ_{rs}, and we are not concerned with any other features of the underlying preference system, let alone taking the trouble to actually determine it. (We do not need to reconstruct the preference system we started with, which we pretend to have forgotten, and in practical situations never even know, to determine the numbers δ_{rs}, in which we are interested.)

With the formulae

$$\hat{X}_r = -(2\hat{M} - 2\varphi_r)^{\frac{1}{2}}, \quad U_r = -\frac{\hat{X}_r}{\lambda_r}$$

and

$$\hat{\delta}_{rs} = \hat{U}_r(\hat{X}_s - \hat{X}_r),$$

and using the numbers λ_r, φ_r and $\hat{M}$ we now have, we find $\hat{\delta}_{rs}$ reproduces the numbers δ_{rs} we determined directly from the preference map with which we started.

The last two examples have been artificially constructed to display the principles and practical working of the calculations. But these calculations have to apply to natural data. We shall use data from Irving Fisher's "The Making of Index Numbers." He uses the data as a basis for the comparison of results obtained using several different formulae and the results here can also enter into this comparison. He gives prices and quantities for 36 commodities in the years 1913-1918.

Example 3. We take Irving Fisher's data for the 36 commodities in the years 1913, 15, 16 and 17. There is absolute consistency, and even median consistency, but not quadratic consistency, since the matrix $G_o'X_o$ has positive determinants. The results of the algorithm cannot therefore be given the strict interpretation we want; but we shall give them nevertheless. The years 1913, 15, 16 and 17 are listed as 0, 1, 2, and 3 .

D	0	1	2	3
0	0	.089	.187	.194
1	-.085	0	.088	.092
2	-.160	-.087	0	.0036
3	-.157	-.085	.0078	0

$\hat{\delta}$	0	1	2	3
0	0	.090	.188	.186
1	-.084	0	.092	.090
2	-.158	-.083	0	.0019
3	-.168	-.087	.00201	0

$\tilde{\delta}$	0	1	2	3
0	0	.082	.152	.151
1	-.093	0	.081	.080
2	-.206	-.096	0	-.0019
3	-.218	-.100	.0020	0

Example 4. The algorithm is now applied just to food, that is to the ten commodities 1, 3, 4, 9, 13, 22, 24, 32, 35, and 36 in the years 1914, 15, 17, and 18, which will be listed as years 0, 1, 2, and 3. Quadratic consistency is verified.

D	0	1	2	3
0	0	.0643	.0720	.2223
1	-.518	0	.0143	.1402
2	-.0495	.0066	0	.1301
3	-.1694	-.1189	-.1151	0

$\hat{\delta}$	0	1	2	3
0	0	.0623	.0666	.2220
1	-.0539	0	.00374	.1383
2	-.0549	-.00356	0	.1280
3	-.1698	-.1222	-.1189	0

$\tilde{\delta}$	0	1	2	3
0	0	.0573	.0609	.1581
1	-.0591	0	.00371	.1041
2	-.0606	-.00358	0	.0968
3	-.2852	-.1819	-.1754	0

Example 5. Fuel, that is the four commodities 7, 8, 10, 23, for the years 1913, 16, 17, and 18 (to be listed as 0, 1, 2, and 3). Quadratic consistency is verified.

D	0	1	2	3
0	0	.161	.331	.395
1	-.126	0	.137	.190
2	-.210	-.109	0	.043
3	-.271	-.160	-.044	0

$\hat{\delta}$	0	1	2	3
0	0	.160	.330	.395
1	-.127	0	.136	.188
2	-.212	-.109	0	.042
3	-.271	-.161	-.045	0

$\tilde{\delta}$	0	1	2	3
0	0	.153	.302	.355
1	-.133	0	.130	.176
2	-.233	-.115	0	.041
3	-.305	-.174	-.046	0

11. Wald's Index-Number Method.

Let φ be a convex quadratic in several variables with gradient g. Then φ attains an absolute maximum, which can be assumed to be 0, at a point c, called its centre, which is also the unique point at which its gradient vanishes. Thus $\varphi(x) \leq 0$; and the condition

$$\varphi(x) = 0 \, , \; g(x) = 0 \, , \; x = c$$

are equivalent. Also

$$\varphi(x) = \tfrac{1}{2}(x-c)'g(x)$$

and

$$(x-c)'g(y) = (y-c)'g(x) \, .$$

Any quadratic which is increasing in any neighbourhood having $g > 0$ (partial derivatives all positive), and has convex levels $\varphi = \varphi_o$, these being surfaces bounding convex regions $\varphi \geq \varphi_o$, has to be a convex function. These general facts about quadratics now will be taken for granted.

Wald [op. cit.] has considered the hypothesis that expenditures are regulated by a quadratic preference function. But a proper preference function must be increasing and have convex levels in the appropriate neighbourhood, otherwise it could not have a unique maximum subject to any expenditure-price restriction; and therefore it must be convex if quadratic. If the prices are p and the expenditure is e, then the chosen composition of goods x such that $p'x = e$ is determined by Gossen's law that preference and price directions coincide in equilibrium,

$$g = p\mu \, ,$$

when $e\mu = x'g$. As e varies with p fixed, the locus of x is a line through c, which is the Engel curve for prices p.

Wald has supposed linear Engel curves for prices p_o, p_1 on two occasions to be given, and has shown how cost of living measurements between the occasions can be made on the hypothesis of a quadratic preference function.

It has to be noticed that, for such a quadratic hypothesis to be admissible on the data, the given Engel curves have to intersect in some point c, which must moreover have a certain location, bounded away from the origin by the price-expenditure constraints; so certainly c cannot be at the origin. Wald's approach, thus qualified, can be greatly simplified; and a formula can be obtained which directly modifies Fisher's formula, in a way which should be considered a correction rather than a generalization.

Suppose the data are that composition x_o, x_1 (vectors of quantities of goods) which are purchased at prices p_o, p_1 (vectors of prices of goods) and that the Engel curves intersect in a point c, so they are the lines joining c to x_o, x_1. With hypothesis of a quadratic φ, Gossen's law gives

$$g(x_o) = p_o\mu_o \ , \ g(x_1) = p_1\mu_1 \ ,$$

for some positive μ_o, μ_1, which, in view of g being the gradient of a quadratic with centre c, have their ratio determined by

$$(x_o-c)'p_1'\mu_1 = (x_1-c)'p_o\mu_o \ ;$$

and they can be chosen otherwise aribtrarily, corresponding to the fact that the same expenditure system is obtained when the preference function is multiplied by an arbitrary constant.

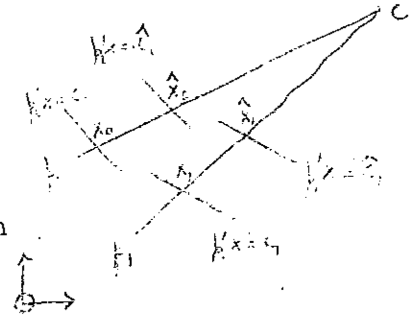
Now

$$\varphi(x_o) = \tfrac{1}{2}(x_o-c)'p_o\mu_o \ , \ \varphi(x_1) = \tfrac{1}{2}(x_1-c)'p_1\mu_1$$

and hence the condition for

$$\varphi(x_o) = \varphi(x_1)$$

is



$$\frac{p_o'(x_o-c)p_1'(x_o-c)}{p_1'(x_1-c)p_o'(x_1-c)} = 1 .$$

In the case $c = 0$, this becomes the Fisher criterion

$$\frac{p_o'x_o p_1'x_o}{p_1'x_1 p_o'x_1} = 1 ,$$

establishing a relation, already well-known [Buschguennce, Konüs, op. cit.], between the Fisher index and a special but improper form of quadratic hypothesis: as has been remarked, this case must be excluded if a quadratic hypothesis is to be properly admissible.

It is now asked, what is the condition for an expenditure $\hat{e}_o$ at prices p_o to have the same value, so far as what can be got with it, as an expenditure $\hat{e}_1$ at prices p_1. The compositions

$$\hat{x}_o = x_o t_o + c(1-t) , \quad \hat{x}_1 = x_1 t_1 + c(1-t_1)$$

with Engel curves for p_o, p_1 such that

$$p_o'\hat{x}_o = \hat{e}_o , \quad p_1'\hat{x}_1 = \hat{e}_1 ,$$

have to satisfy $\varphi(\hat{x}_o) = \varphi(\hat{x}_1)$, or the equivalent conditions just deduced. Apparently,

$$t_o = \frac{\hat{e}_o - p_o'c}{p_o'(x_o-c)} , \quad t_1 = \frac{\hat{e}_1 - p_1'c}{p_1'(x_1-c)}$$

so that

$$\hat{x}_o - c = (x_o-c)\frac{\hat{e}_o - p_o'c}{p_o'(x_o-c)} , \quad \hat{x}_1 - c = \frac{\hat{e}_1 - p_1'c}{p_1'(x_1-c)}$$

and hence this condition is

$$\left(\frac{\hat{e}_o - p_o'c}{\hat{e}_1 - p_1'c}\right)^2 = \frac{p_o'(x_o-c)p_o'(x_1-c)}{p_1'(x_1-c)p_1'(x_o-c)} ,$$

reducing to the condition in the form already found in the case $\hat{e}_o = p_o'x_o$, $\hat{e}_1 = p_1'x_1$.

This formula determines the expenditure $\hat{e}_1$ at prices p_1 which is equivalent to any expenditure $\hat{e}_o$ at prices p_o ; or, equivalently, it determines the multiplier $\rho_{o1} = \frac{\hat{e}_1}{\hat{e}_o} = \rho_{o1}(\hat{e}_o)$ of the expenditure $\hat{e}_o$ which will compensate it for the price change from p_o to p_1 . This multiplier defines the cost-of-living index for base and object occasions with prices p_o and p_1 , corresponding to any level of expenditure $\hat{e}_o$. It thus determines a cost of living index corresponding to every standard of living, on a preference hypothesis which is both of the proper form and admissible on the data. The preference hypothesis has been further restricted to be of the special form which is represented by a quadratic preference function in a certain neighbourhood containing x_o , x_1 . In fact, if any, an infinite class of such hypotheses has to be admitted, if there are more than two goods, but each of these hypotheses leads to the same result for the wanted cost of living measurement, and therefore the choice between them is immaterial, so far as this measurement is concerned.

Naturally, the rather complicated quadratic interval algorithm already given for expenditure data of four occasions applies to Wald's case, since this is the special case in which the four price-vectors are parallel in two pairs. But peculiarly simple formulae go with this special case, so the more general formulae are not needed.

Three linear Engel curves, through points x_o , x_1 , x_2 , with corresponding prices p_o , p_1 , p_2 , intersecting in pairs and which are not coplanar must be concurrent in a single point c ; but still the quadratic hypothesis will not be admissible unless

$$\frac{p_o'(x_1-c)}{p_1'(x_o-c)} \; \frac{p_1'(x_2-c)}{p_2'(x_1-c)} \; \frac{p_2'(x_o-c)}{p_o'(x_2-c)} = 1 .$$

Given any number of such curves, this condition applied to every three is necessary and sufficient for the admissibility of the quadratic hypothesis on the data they provide; so it is generally not admissible for more than two taken together. Nevertheless, the quadratic model can still be appropriate for analysis of data which are not exactly consistent on that model, though this makes for a statistical kind of analysis, and gives rise to a more elaborate algebraical formalism.

The extreme simplicity of the formulae here derived for Wald's case, and the fact that an intersecting pair of linear Engel curves gives a convenient statistical model for expenditure data in two periods, is a recommendation for practical usefulness in filling the same role, in the same two-period framework, as the conventionally used but conceptually inadequate Laspeyres index. However, when analysis has to span several years, such as would be the case in insurance policies based on "real-value"--then the analysis should be still more broadly based, and give proper recognition to all available data, which, together with the data in the base and object years, are all equally relevant to preference structure.

Bowley has given a method of approximate index-number construction based on a quadratic preference hypothesis. The method of approximation has been criticized by Frisch[1], who has proposed another construction, also based on quadratics, known as the "double-expenditure method."[2] This method also relies on an approximation, the validity of which has been criticized by Wald [op. cit.]. That criticism can be linked to the criticism made here of Fisher's formula. However, the

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[1] Ragnar Frisch, "The Problem of Index Numbers," Econometrica, Vol. 4 (January, 1936), pp. 1-38.

[2] Ibid. "The Double-Expenditure Method," Econometrica, Vol. 6 (January, 1938), p. 85.

method based on quadratics which Wald proposes, when taken together with the further qualifications made here, is exact. Without these qualifications, it can only be valid in some approximate sense, since then no proper quadratic hypotheses which are exactly admissible on the data will exist.

PREVIOUS RESEARCH MEMORANDA

"Beneficiation of United States Manganese Reserves"
Research Memorandum No. 1 (January 1957).

"Programming the Supply of Strategic Materials."
Research Memorandum No. 2 (September 1957).

"Econometric Analysis of the United States Manganese Problem."
Final Report, Part I, Research Memorandum No. 3 (October 1958).

"On the Magnification of Derived Demand." R. G. Davis and W. G. Mellon.
Research Memorandum No. 4 (November 1958).

"A General Theory of Measurement-Applications to Utility." J. Pfanzagl.
Research Memorandum No. 5 (December 1958).

"Estimates of National and Regional Steel Production in the United States for 1970." R. E. Kuenne. Research Memorandum No. 6 (February 1959).

"Value and Expenditure." S. N. Afriat.
Research Memorandum No. 7 (February 1959).

"Structures in Logistics Operations." H. D. Mills.
Research Memorandum No. 8 (February 1959).

"Stochastic Properties of Elementary Logistic Components." H. D. Mills
Research Memorandum No. 9 (February 1959).

"Uncertainty and the Expansion of Bank Credit." Daniel Orr and Walter G. Mellon. Research Memorandum No. 10 (November 1959).

"The Concepts of Equivalence, Order and Scale in the Analysis of Preferences." S. N. Afriat. Research Memorandum No. 11 (January 1960).

"First Report of the Time Series Project." Clive W. J. Granger.
Research Memorandum No. 12 (February 1960).

"The Axiomatic Concept of a Scale and the Logic of Preferences in the Theory of Consumers' Expenditures." S. N. Afriat.
Research Memorandum No. 13 (March 1960).

"Econometric Analysis of the United States Manganese Problem." Final Report, Part II, Research Memorandum No. 14 (March 1960).

"Production Stability and Inventory Variation." Daniel Orr.
Research Memorandum No. 15 (May 1960).

"Statistical Projection." S. N. Afriat.
Research Memorandum No. 16 (June 1960).

"Preference Structures and the Measurement of Military Worth." Richard G. Davis. Research Memorandum No. 17 (September 1960).

"The System of Inequalities $a_{rs} > X_r - X_s$." S. N. Afriat. Research Memorandum No. 18 (October 1960).

"A New Look at Economic Time Series Analysis." Oskar Morgenstern. Research Memorandum No. 19 (January 1961).

"Gradient Configurations and Quadratic Functions." S. N. Afriat. Research Memorandum No. 20 (January 1961).

"Expenditure Configurations." S. N. Afriat. Research Memorandum No. 21 (February 1961).

"Subjective Programming." Robert J. Aumann. Research Memorandum No. 22 (February 1961).

"Almost Strictly Competitive Games." Robert J. Aumann. Research Memorandum No. 23 (February 1961).

"The Cost-of-Living Index: Algebraical Theory." S. N. Afriat. Research Memorandum No. 24 (March 1961).

"The n-Person Bargaining Game." Koichi Miyasawa. Research Memorandum No. 25 (March 1961).

"Utility Theory Without the Completeness Axiom." Robert J. Aumann. Research Memorandum No. 26 (April 1961).

"The Cost-of-Living Index: Combinatorial Theory." S. N. Afriat. Research Memorandum No. 27 (April 1961).

"Competition Through the Introduction of New Products." Robert Reichardt. Research Memorandum No. 28 (July 1961).